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SELF-SIMILAR PROBLEM OF IMPACT LOADING OF AN ELASTIC HALF-SPACE

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A self-similar dynamic problem of the nonlinear theory of elasticity concerned with an oblique impact on a half-space, is considered. Thermodynamic restrictions affecting the existence of shock waves in an elastic medium are used as the basis for constructing uniquely possible combinations of wave fronts, depending on the boundary conditions, by means of which the perturbations are propagated through the medium. Particular features of the numerical solution of the problem are discussed and its results given. Plane self-similar problems of the nonlinear dynamic theory of elasticity have been studied before [1—3]. Qualitative features of problems belonging to the class in question were discussed in [1] and the laws of thermodynamics were suggested as the basis for obtaining a uniquely possible wave pattern. Problems of reflection of plane shock waves from the plane barriers were considered in [2], while an analytic solution of the problem in question was constructed in [3] for the case of small boundary perturbations, for a neo—Hookean model of elastic medium. Below the problem of impact loading of a half-space with the boundary conditions given in terms of velocities is solved within the framework of the quadratic theory of elasticity.

1. A system of equations of dynamic deformation of elastic medium written in the Cartesian rectangular coordinate system has the form [4]

$$\sigma_{ij} = \frac{\rho}{\rho_0}\frac{\partial W}{\partial e_{ik}}(\delta_{kj} - 2e_{kj}), \quad \sigma_{ij,\,j} = \rho\frac{\partial v_i}{\partial t} \tag{1.1}$$

$$v_i = \partial u_i/\partial t + v_j u_{i,\,j}, \quad 2e_{ij} = u_{i,\,j} + u_{j,\,i} - u_{k,\,i}u_{k,\,j}$$

$$I_1 = e_{jj}, \quad I_2 = e_{ij}e_{ji}, \quad I_3 = e_{ij}e_{jk}e_{ki}$$

$$\rho/\rho_0 = (1 - 2I_1 + 2I_1^2 - 2I_2 - {}^4/_3 I_1^3 + 4I_1I_2 - {}^8/_3 I_3)^{1/2}$$

Here $\sigma_{ij}, e_{ij}, v_i, u_i$ denote the components of the stress tensors, finite Almansi deformations, the velocity and displacement vectors respectively, while ρ and ρ_0 denote the density in the stressed and unstressed state. We close the system of equations (1.1) by employing, in what follows, the relations connecting the elastic potential W with the invariants of the Almansi deformation tensor

$$W = {}^1/_2\lambda I_1^2 + \mu I_2 + lI_1I_2 + mI_1^3 + nI_3 \tag{1.2}$$

Let, beginning from the instant $t = 0$, load the boundary $x_1 > 0$ of the half-space the elastic properties of which are described by (1.2), in such a manner that every material point of the boundary begins to move with constant velocity the components of which are v_{10}, v_{20} and $v_{30} = 0$. Restricting ourselves to the second order terms in the components of the tensor $u_{i,j}$, we can reduce the system (1.1), (1.2) in the present case to the form

$$(\lambda + 2\mu)u_{1,11} + 2\alpha u_{1,1} u_{1,11} + 2\gamma u_{2,1}u_{2,11} = \rho_0 (1 - u_{1,1})dv_1 / dt \qquad (1.3)$$

$$\mu u_{2,11} + (2\gamma - \mu)(u_{2,1}u_{1,11} + u_{1,1}u_{2,11}) = \rho_0 (1 - u_{1,1})dv_2 / dt$$

$$\alpha = 3(l + m + n) - {}^7/_2 (\lambda + 2\mu), \quad \gamma = {}^1/_2 l + {}^3/_4 n - {}^1/_2 \lambda - \mu$$

Let us introduce the self-similar variable and rewrite (1.3) in the following dimensionless form:

$$AT'' + a_2\Theta'\Theta'' = 0, \quad (a_3 - \xi^2)\Theta' T'' + B\Theta'' = 0 \qquad (1.4)$$

$$\xi = x_1 / (G_0 t), \qquad u_1 = G_0 t T(\xi), \quad u_2 = G_0 t \Theta(\xi), \quad G_0 = [(\lambda + 2\mu) / \rho_0]^{1/2}$$

$$a_1 = 2\alpha / (\lambda + 2\mu), \quad a_2 = 2\gamma / (\lambda + 2\mu), \quad a_3 = (2\gamma - \mu) / (\lambda + 2\mu)$$

$$A = 1 - \xi^2 + (a_1 - 2)T' - 2\xi T, \quad \varkappa = \mu / (\lambda + 2\mu)$$

$$B = \varkappa - \xi^2 + (a_3 - 2\varkappa)T' - \xi T' + 2\xi T$$

The relations (1.4) in which a prime denotes differentiation with respect to ξ, represent a homogeneous system of two, second order ordinary differential equations.

Let us consider the one-dimensional problem by putting $v_{20} = 0$. Then (1.4) shows that a nontrivial solution is possible only when $A = 0$. The last equation is linear, and its solution has the form

$$T(\xi) = \exp(\xi^2 a) F(\xi), \quad F(\xi) = \int_1^{\xi} (\eta^2 - 1)\, a \exp(-\eta^2 a)\, d\eta \qquad (1.5)$$

$$v_1 = G_0 \frac{[\exp(\xi^2 a) F(\xi)(1 - 2\xi^2 a) - \xi a(\xi^2 - 1)}{1 - a(\xi^2 - 1) - \exp(\xi^2 a) 2\xi a F(\xi)}, \quad a = (a_1 - 2)^{-1}$$

We assume in (1.5) that the leading front of the perturbation occupies the position $\xi = 1$ (this follows directly from the equation $A = 0$). The nontrivial solution should hold on the interval $\xi_0 \leqslant \xi \leqslant 1$ where $\xi_0 \leqslant v_{10} / G_0$ and $v_1(\xi_0) = v_{10}$. Substituting the latter relations into (1.5) we find, that if $a_1 < 2$, then the nontrivial solution will be possible only when $v_{10} < 0$. For real materials [5] $a_1 \leqslant -3.5$, i.e. we arrive at the same results as in the case of a real gas. A centralized Riemann wave appears only when the action exerted upon the medium causes its expansion ($v_{10} < 0$). A perturbation leading to compression ($v_{10} > 0$) propagates through the medium in the form of a shock wave. In this case we have $T'' = 0$ everywhere except in the case $\xi = \xi^* = G / G_0$ where G is the velocity of the shock wave obtained from the equation $A = 0$, when $T = 0$. We note that ξ^* varies with the intensity of the shock wave but $\xi^* > 1$ everywhere, i.e. $G > G_0$.

If $v_{20} \neq 0$, then nontrivial solutions are possible when the determinant of the system (1.4) is zero

$$AB - a_2 (a_3 - \xi^2)\Theta'^2 = 0 \qquad (1.6)$$

In the regions in which (1.6) does not hold, (1.4) has a trivial solution

$$T' = \text{const}, \quad \Theta' = \text{const}.$$

In the regions where (1.6) holds, the solution is obtained from the centralized Riemann wave the leading and trailing fronts of which are themselves weak waves. Apart from the centralized waves, a perturbation may propagate through the medium in the form of shock waves. We obtain the actual wave pattern using the fact that the number of boundary conditions is equal to the number of the constants of integration of the system (1.4).

Let n shock waves and k centralized waves ($2k$ weak waves) propagate through the medium as a result of a dynamic action. Four integration constants must be determined in the zone of the trivial solution of (1.4) and three constants in the zone of the nontrivial solution, while the velocities (position) of the shock waves and weak waves remain unknown. In the present case we have $n + k$ zones in which the solution of (1.4) is trivial, and k zones in which it is nontrivial, therefore the total number of integration constants is equal to $4(n + k) + 3k + 2k + n$. Four boundary conditions can be specified at each surface of discontinuity. In the case of a shock wave these conditions will be $[u_i] = 0$ (continuity of the displacements) and $[\sigma_{ij}]\nu_j = \rho^+ (v_n^+ - G)[v_i]$ (conservation of momentum) and for a weak wave we have $[v_i] = 0$ and the corresponding dynamic condition of compatibility of the first order discontinuities [6]. Thus we can equate the number of the boundary conditions with that of the integration constants and write $5n + 9k = 4(n + 2k) + 2$ from which follows $n + k = 2$.

An oblique impact acting on an elastic half-space generates in it either two shock waves, or two centralized Riemann waves, or one shock wave and one centralized wave (four cases). The choice of one of the four possible wave patterns for each particular case, the choice depending on the type of the boundary conditions and of the system (1.4), is not possible as it was in the case of the one-dimensional problem, because of the complexity of the patterns. A different approach is based on the study of the properties of the shock waves in an elastic medium subjected to a finite, plane deformation.

2. We shall represent a shock wave propagating through an elastic medium the motion of which is described by (1.1), (1.2), by a surface on which the displacements are continuous, while the components of the displacement gradient, deformation and stress tensors and of the velocity vector, and the density, all exhibit a first order discontinuity. If the x_1-axis of the moving x_1, x_2-coordinate system attached to the surface of discontinuity is normal to this surface, then the components of the velocity vector are given by formulas

$$v_i = \delta u_i / \delta t + (v_1 - G)u_{i,1} + v_2 u_{i,2}, \quad i = 1, 2 \tag{2.1}$$

Applying to (2.1) the discretizing operation and solving the resulting relations for $[v_1]$ and $[v_2]$, we obtain

$$[v_1] = v_1^+ - v_1^- = (v_1^- - G)\omega^{-1}(p_2\tau_1 + u_{1,2}\tau_2), \quad [v_2] = (v_1^- - G)\omega^{-1}(p_1\tau_2 + u_{2,1}\tau_1) \tag{2.2}$$

$$p_1 = 1 - u_{1,1}, \quad p_2 = 1 - u_{2,2}, \quad [u_{i,j}] = \tau_i\delta_{j1}\,[\delta u_i / \delta t] = 0$$
$$\omega = p_1p_2 - u_{1,2}u_{2,1}$$

The plus and minus superscripts in (2.2) denote the quantities computed in front and directly behind the shock wave, respectively. The plus sign is omitted from the components of the tensor $u_{i,j}$ since in what follows they will always be computed in front of the surface of discontinuity. Substituting (2.2), (1.1) and (1.2) written in terms of the discontinuities into the dynamic conditions of compatibility of the discontinuities, leads to a system of two equations from three unknowns τ_1, τ_2 and V

$$(V - s_1)\tau_1 + R_1\tau_2 + (\gamma - \mu)\tau_2^2 = 0 \qquad (2.3)$$
$$(V - s_2)\tau_2 + R_2\tau_1 = 0$$
$$V = \rho^-(v_1^- - G)^2, \quad s_1 = p_1(\lambda + 2\mu) + 2\alpha u_{1,1} + \beta u_{2,2} - \alpha\tau_1$$
$$s_2 = p_2\mu + 2\gamma(u_{1,1} + u_{2,2}) - 2\gamma\tau_1, \quad \beta = 6m + 2l - 4\lambda - 2\mu$$
$$R_1 = b_1V - k_1, \quad R_2 = b_2V - k_2, \quad k_1 = (2\gamma + \lambda)u_{1,2} + 2(\gamma - \mu)u_{2,1}$$
$$k_2 = (2\gamma - \mu)u_{1,2} + 2\gamma u_{2,1}, \quad b_1 = u_{1,2}, \quad b_2 = u_{2,1}$$

In (2.3) the deformed state in front of the surface of discontinuity is assumed given, and the parameter V characterizes the rate of propagation of the shock wave. Assuming that τ_1 is known, we arrive in accordance with (2.3) to the following cubic equation in V:

$$(V - s_1)(V - s_2)^2 + R_1R_2(V - s_2) + (\gamma - \mu)R_2^2\tau_1 = 0 \qquad (2.4)$$

When $u_{1,2} = u_{2,1} = 0$, $V_1 = s_1$, $V_2 = V_3 = s_2$, the solution of (2.4) is the simplest. Substitution of the first root into (2.3) yields $\tau_2 = 0$, i.e. the shock wave is longitudinal. Since $s_1 > s_2$, it follows that in the case when the medium is undeformed or subjected to hydrostatic compression, the leading front of the perturbation propagating through the medium will, as long as it is a shock wave, only be a longitudinal shock wave. When the influence of the nonlinearities diminishes, the velocity of the longitudinal shock wave will tend to the value G_0, the latter denoting the velocity of the irrotational shock wave in the linear theory of elasticity. The value of the coincident roots corresponds to a shock wave on which $\tau_2 \neq 0$ and $\tau_1 \neq 0$. When the influence of the nonlinearities diminishes, the velocity of this wave tends to the value $\{\mu / \rho_0\}^{1/2}$ of the velocity of the equivoluminal shock wave in the linear theory of elasticity. From now on, we shall call this shock wave quasi-transverse. If $u_{1,2}$ or $u_{2,1}$ is not zero, then the first shock wave will not be strictly longitudinal, i.e. $\tau_2 \neq 0$ on this wave, and we shall call this wave quasi-longitudinal.

When shear deformations are present in front of the surface of discontinuity, then the velocities of the quasi-longitudinal and quasi-transverse waves are given by the following corresponding expressions:

$$V_1 = s_1 + R_{11}R_{21} / (s_1 - s_2) + [(\gamma - \mu)R_{21} / (s_1 - s_2)]^2 \qquad (2.5)$$
$$V_{2,3} = s_2 \pm \{R_{1,2}R_{2,2} - (\gamma - \mu)R_{22}^2 / (s_1 - s_2)\}^{1/2}$$
$$(R_{ij} = b_is_j - k_j, \quad i, j = 1, 2)$$

On substituting the second relation of (2.5) into (2.3) we find that the quasi-transverse shock waves are possible only when the following condition holds:

$$(s_1 - s_2)\tau_1 / (\gamma - \mu) \geqslant 0 \qquad (2.6)$$

For real materials we have $\gamma < 0$, consequently $\tau_1 \leqslant 0$, i.e. the quasi-transverse shock wave is at the same time a rarefaction wave. The components of the displacement gradient tensor $u_{i,j}$ are assumed small (quadratic theory of elasticity), therefore we must assume that on the quasi-transverse shock wave τ_1 is of the second order of smallness compared with τ_2.

Omitting cumbersome transformations connected with determining the discontinuities according to (1.1), (1.2) and (2.2) we can write, for the thermodynamic conditions of compatibility of the discontinuities [7]

$$-{}^1/_2 c\,[v_j][v_j] + \sigma_{j1}\,[v_j] - [W]c\,/\,\rho_0 \geqslant 0, \quad c = V\,/\,(v_1^- - G) \qquad (2.7)$$

the following final result for the case of a quasi-longitudinal wave when the parameter V is given by the first equation of (2.5):

$$(l + m + n - {}^3/_2\lambda - 3\mu)\tau_1^3 \leqslant 0 \qquad (2.8)$$

We note that the deformed state in front of the surface of discontinuity does not affect the magnitude of the entropy jump at the quasi-longitudinal shock wave. If l, m and n negative [5] or if their order is lower than that of λ and μ, then (2.8) yields the analog of the Cemplen theorem for the perfect gas, i.e. the only quasi-longitudinal low intensity shock waves possible in an elastic medium are those, which lead to compression of the medium. If the boundary conditions are such that a rarefaction occurs, then the shock wave representing the solution of the corresponding linear problem must be replaced, in the nonlinear case, by the centralized Riemann rarefaction wave. We arrive at the same result by basing our arguments on the one-dimensional case discussed earlier where the solution obtained without using any thermodynamic constraints was nevertheless in complete agreement with them.

In the case of a quasi-transverse shock wave when V is determined from the second equation of (2.5), the inequality (2.7) becomes an identity to within the third order terms in $u_{i,j}$. It follows that in (1.2) the fourth order terms in $u_{i,j}$ must be taken into account. If we restrict ourselves in (1.2) to terms which have been written out, then from (2.7) it follows that quasi-transverse, low intensity shock waves are impossible in the elastic medium. Numerical computations show that the centalized shear wave which represents a nontrivial solution of the system (1.4) leads, just as the quasi-transverse shock wave (2.6), to a dilation of the medium. Using the principle of least work done by the entropy, we conclude that the quasi-transverse shock waves are not thermodynamically feasible. It is this particular fact which explains the experimental data which indicate the impossibility of registering transverse shock waves at considerable distances from their sources.

3. Let $v_{10} \geqslant 0$, i.e. assume that the boundary conditions of the problem lead to compression of the medium. Itilizing the previous arguments we conclude, that a perturbation propagates through the medium in the form of a longitudinal ($\tau = 0$ since $u_{1,2} = u_{2,1} = 0$ in front of the surface of discontinuity), shock and centralized shear waves. Let us denote by $\xi^* = G\,/\,G_0 = 1 + \delta$ the position (Fig. 1) of the shock wave and by ξ_0 and ξ_1 the leading and trailing front of the centralized wave. Then we have in the zone between ξ^* and ξ_0 $\Theta = 0, T' = \text{const}$, consequently the only values constant and different from zero will be σ_{11}, e_{11} and v_1. Using the dynamic and kinematic conditions of compatibility of the discontinuties at

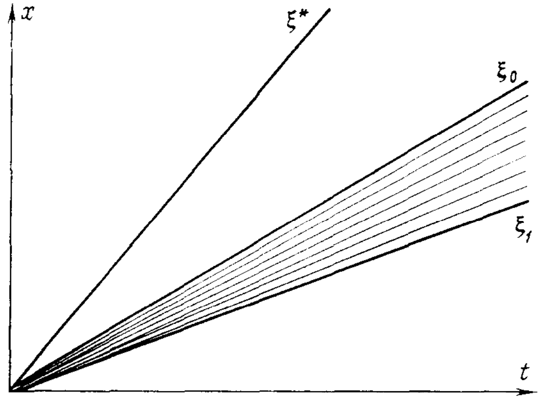

Fig. 1

the longitudinal shock wave

$$[\sigma_{11}] = -\rho_0 G [v_1], \quad [\partial u_1 / \partial t] = -G\tau_1$$

and the relations (1.1), (1.2) and (2.2), we obtain

$$v_1 = -GT_0' / (1 - T_0'), \quad \sigma_{11} = (\lambda + 2\mu + \alpha T_0')T_0' \qquad (3.1)$$
$$(1 + a_1 T_0')(1 - T_0') = (1 + \delta)^2$$

According to (3.1), all parameters defining the deformed state and stress state can be expressed, in the zone contained between ξ^* and ξ_0, by a single parameter such as e.g. δ. Within the zone contained between ξ_0 and ξ_1, the solution of the problem can be obtained from the following system of ordinary differential equations derived from (1.4) and (1.6):

$$\Theta' = \left\{\frac{AB}{a_2 f}\right\}^{1/2} \qquad (3.2)$$
$$T'' = 2\,\frac{AB\xi - Bf(\xi + T - \xi T') - Af(\xi - T)}{2Af^2 - Bf(a_1 - 2) - Af(f - 2\varkappa)}, \quad f = a_3 - \xi^2$$

When solving the problem by numerical methods, we must remember that $\Theta''(\xi_0) = \infty$ and that at the leading front of the centralized rarefaction wave $T'' = \infty$.

The boundary conditions for (3.2) are given by

$$\Theta(\xi_0) = 0, \quad T'(\xi_0) = T_0', \quad T(\xi_0) = T_0'(\xi_0 - \xi^*) + T(\xi^*), \quad T(\xi^*) = 0 \qquad (3.3)$$
$$R = v_{10} p, \quad p = 1 - T'(\xi_1), \quad R = T(\xi_1) - \xi_1 T'(\xi_1),$$
$$\Theta(\xi_1) - \xi_1 \Theta'(\xi_1) + R p^{-1} \Theta'(\xi_1) = v_{20}$$

The quantity ξ_0 is given in terms of δ by the equation $B(\xi_0) = 0$. It follows that we must solve the boundary value problem for (3.2) with boundary conditions (3.3), in the zone contained between ξ_0 and ξ_1. The fact that there are five boundary conditions in (3.3) and not three, which would agree with the order of the system of equations (3.2), is explained by the fact that ξ_1 and δ in (3.3) remain unknown and have to be determined in the course of solving the problem.

The problem was solved numerically, and the results obtained confirmed the correctness of the choice of the wave front configuarations.

Certain of the results obtained in the course of solving the problem of pure shear when $v_{20} \neq 0$ and $\sigma_{11} = 0$ at the boundary of the half-space, are given below

$\frac{v_{20}}{G} \cdot 10^5$	152	401	569	681	1190	1631	2249
$(\eta_0 - 1) \cdot 10^5$	1	7	14	21	63	115	662
$(1 - \eta_1) \cdot 10^5$	1	7	15	21	68	134	289
$\left(\frac{G}{G_0} - 1\right) \cdot 10^5$	1	7	14	20	60	110	200
$h \cdot 10^5$	50	357	703	1013	3243	6533	9647
$-\frac{\sigma_{11}^0}{\lambda + 2\mu} \cdot 10^5$	1	7	15	21	63	117	684
$-\frac{\sigma_{12}^1}{\lambda + 2\mu} \cdot 10^5$	82	218	308	369	639	864	3247

where the subscript 0 refers to the zone between ξ^* and ξ_0, and the subscript 1 to the zone between ξ_1 and the boundary of the half-space $\eta = \rho / \rho_0$, $h = \xi_0 - \xi_1$. It can be seen that the process of deformation takes place as follows: the leading front of the perturbation is a compression shock wave (Poynting effect which states that pure shear causes a compression of the medium) so that in the zone 0 the medium is compressed, while the centralized shear wave leads, at the same time, to a dilatation of the medium. Finally, the Weissenberg effect is observed in zone 1 (pure shear leads to the dilatation of the material). The width h of the centalized wave, the absolute values of the Poynting and Weissenberg effects and the intensity of the shock wave all increase with the increasing action (v_{20} / G_0) on the medium. The stress σ_{11} decreases in the zone of nontrivial solution (between ξ_0 and ξ_1) with decreasing ξ, and this is also true when $v_{10} > 0$ and the quantity σ_{12} increases.

All functions are monotonous in the zone of nontrivial solution. The position of the leading front of the centalized wave for $a_1 = -36.42$, $a_2 = -5.39$, $a_3 = -5.69$, $\varkappa = 0.29$ (which corresponds [5] to steel and was used to obtain the data tabulated) varies within the limits $0.5439 \leqslant \xi_0 \leqslant 0.5566$ when v_{20} / G varies within the limits shown in the table. At the same time ξ_0 increases with increasing action. In the case of an oblique impact $(v_{10} > 0)$, ξ_0 and the width of the centalized wave both vary within approximately the same range.

If $v_{10} < 0$, then the compressive shock wave must be replaced by the centalized rarefaction wave (the zone between $\xi_1 = 1$ and ξ_2 shown in Fig. 2). Equation $\xi_1 = 1$ follows directly from (1.6) if we put $T' = 0$ and $\Theta' = 0$. In the region of the centralized rarefaction wave $\Theta \equiv 0$ and this follows directly from (1.4) and (1.6), i.e. the centralized rarefaction wave is strictly longitudinal (the shear deformations propagate more slowly than the volume deformations). The function T can be found from the first order differential equations $A = 0$ with the boundary condition $T(1) = 0$. In the zone contained between ξ_3 and ξ_4 (centralized shear wave) the solution is obtained from the system (3.2) with the boundary conditions of the form

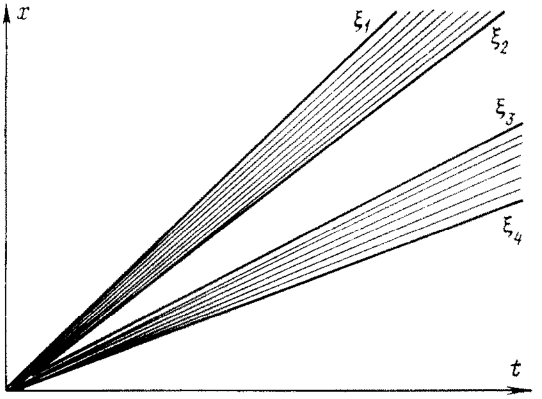

Fig. 2

(3.3) where ξ^* must be replaced by ξ_2, ξ_0 by ξ_3 and ξ_1 by ξ_4. The five boundary conditions (3.3) are used to determine three constants of integration of the system (3.2), and the values of ξ_2 and ξ_4. The quantity ξ_3 is found from the condition $B(\xi_3) = 0$. In solving the problem numerically it was found that T'' becomes infinite when $\xi = 1$. The behavior of all functions in the zone of the centralized waves in monotonous, and the properties of the equations are such that σ_{11} and v_2 increase, while σ_{12}, ρ and v_1 decrease with decreasing ξ just as in the case discussed above.

REFERENCES

1. Burenin, A.A., Lapygin, V. V. and Chernyshov, A. D. On solving plane self-similar problems of the nonlinear dynamic theory of elasticity. Materials of the Symposium "Nonlinear Deformation Waves", pt. 2, Tallin 1978.

2. Wright, T. W. Reflection of oblique shock waves in elastic solids. Internat. J. Solids and Structure, Vol. 7, No. 2, 1971.

3. Chernykh, E. M. Self-similar problem of impact loading in a nonlinearly elastic material. PMM Vol. 31, No. 5, 1967.

4. Sedov, L. I. Introduction to the Mechanics of a Continuum. Moscow, Fizmatgiz, 1962. (see also English translation, Pergamon Press, Book №. 09878, 1965).

5. Zarembo, B. K. and Krasil'nikov, V. A. Introduction to Nonlinear Acoustics. Moscow, "Nauka", 1966.

6. Thomas, T., Plastic Flow and Fracture in Solids. Mathematics in Science and Engineering, Vol. 2, New York—London, Academy Press Inc., 1961.

7. Chernyshov, A. D. On the conditions of propagation of shock waves in the media with elastic and plastic properties. In coll. Controversial Problems of the Mechanics of Rocks. Alma—Ata, "Nauka", 1972.

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